Lecture 2

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY MSc

Subject : Financial Mathematics

Subject Code: PPSAS102

Chapter: Unit 1 – Chapter 2

Chapter Name: Measurement of Interest

Today's Agenda

1. Interest
1. Measures of Interest Rates
1. Fund Value
1. Principle of Consistency
1. Effective Rate of Interest
1. Simple Interest
1. Compound Interest
1. Present Value
9. Effective Rate of Discount
9. Equivalent Rates
9. Simple Discount
9. Compound Discount
9. Useful Insights
9. Nominal Rates of Interest
9. Force of Interest

0

Introduction

What is Interest? (In financial terms)

1 Interest

Interest – **Compensation that a borrowers pays for use of capital.**

Form of a rent that the borrower pays to the lender to compensate for the **loss of use of the capital** by the lender while it is loaned to the borrower.

1.1 Rationale

- **Investment opportunity theory/ Economic productivity of capital** – if you borrow money to run a successful business, the borrowed money allows you to produce more money and you should assign some of that gain to the lender (Good faith)

- **Time preference theory** – people prefer to have money now rather than the same amount of money at some later date. If you lend it, you no longer have the option of immediately using your money. Interest compensates a lender for this loss of choice.

- **Default risk** – lender should be compensated for the possibility that the borrower defaults and the capital is lost.

interest

Russia

In the real world, investments have an element of risk and investors sometimes lose money.

1.2 Importance of Interest

The Reserve Bank of India sets the “ repo rates” and “ reverse repo rates”, a target rate at which banks can borrow and invest funds with one another. This rate affects the more general cost of borrowing and also has an effect on the **stock** and **bond markets**.

Higher interest rates tend to reduce the value of other investments.

Irrational Exuberance

After the close of trading on North American financial markets on Thursday, December 5, 1996, Federal Reserve Board chairman Alan Greenspan delivered a lecture at The American Enterprise Institute for Public Policy Research.

In that speech, Mr Greenspan commented on the possible negative consequences of “irrational exuberance” in the financial markets.

The speech was widely interpreted by investment traders as indicating that stocks in the US market were overvalued and that the Federal Reserve Board might increase US interest rates, which might affect interest rates worldwide.

Although US markets had already closed, those in the Far East were just opening for trading on December 6, 1996. Japan’s main stock market index dropped 3.2%, the Hong Kong stock market dropped almost 3%. As the opening of trading in the various world markets moved westward throughout the day, market drops continued to occur. The German market fell 4% and the London market fell 2%. When the New York Stock Exchange opened at 9:30 AM EST on Friday, December 6, 1996, it dropped about 2% in the first 30 minutes of trading, although the market did recover later in the day.

Sources: www.federalreserve.gov , www.pbs.org/newshour/bb/economy/december96/greenspan_12-6.html

Key Rates & Mortgage rates (%)

	Current	1 month prior	3 months prior	6 months prior	1 year prior
RBI Target rate (Repo Rate)	4.00	4.00	4.40	5.15	5.40
3-month LIBOR	0.25	0.25	0.34	1.25	2.08
Prime Rate	3.25	3.25	3.25	4.25	5.00
15-year Mortgage	2.42	2.44	2.62	2.75	3.00
30-year Mortgage	2.93	2.88	3.18	3.50	3.60

7% - 8%

Rates are taken as at 15 September, 2020.

2 Measures of Interest Rates

Commonly used growth patterns for investment – simple and compound interest.

Alternative measures:

- Nominal annual rate of interest
- Rate of discount – Simple and compound
- Force of interest

$\boxed{P \times R \times T}$

dt

2.1 Interest Accumulation

$A(t_1, t_2)$

$A(n) \quad A(t_1, t_1)$

$A(2, 2) =$

$A(0) =$

Common financial transaction – investment of an amount of money at interest.

- Initial amount of money (capital) invested – **Principal**
- Total amount received after a period of time – **Accumulated value**
- Difference between the accumulated value and the principal – **Amount of interest/ interest**, earned during the period of investment

The unit in which time is measured is called the measurement period, or just period.
The most common measurement period is one year, and this will be assumed unless stated otherwise.

$I = y - x \qquad \frac{120}{100} = 1.2$

100 (at 0) → 120 (at 1)

Principal: x (at 0) → y — Accumul value

100 (at 0)

$A(0) = \frac{100}{100} = 1$

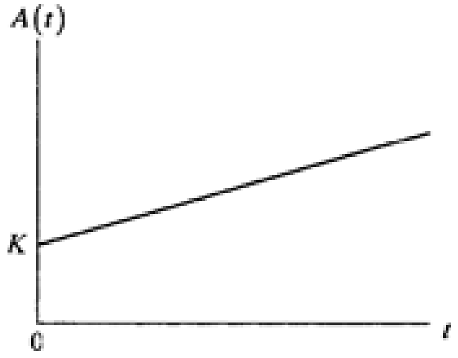
2.2 Accumulation Factor

- Consider an investment of one unit of principal.
- Accumulating factor $A(t)$ gives the accumulated value at time $t \geq 0$ of an original investment of 1.
- $A(0) = 1$ i.e. accumulated value at the time of investment is amount invested, which is 1.
- If C is the amount invested at time 0, accumulated value after t time is $C.A(t)$
- $A(t)$ is generally an increasing function.
- A decrease in the functional values for increasing t would imply negative interest which is very rare.
- If interest accrues continuously, as is usually the case, the function will be continuous.

$A(t)$.

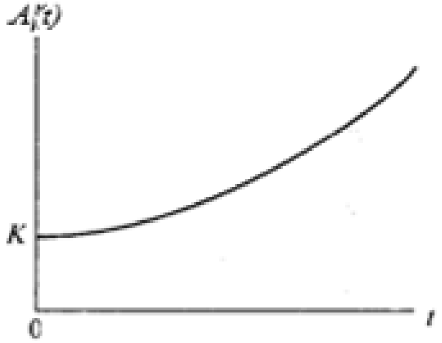
$C \times A(t)$.

2.3 Types of Accumulation Factor

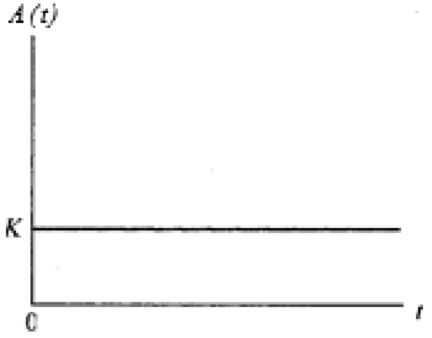


A. Linear accumulation

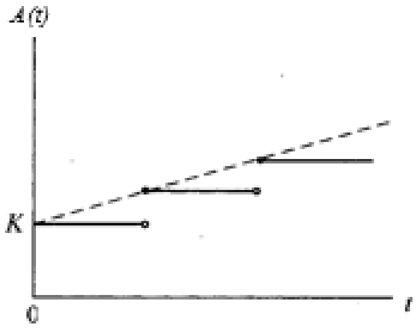
simple int



B. Non-linear accumulation, in this case an exponential curve.



C. An accumulation factor which is horizontal, i.e. the slope is zero.
This figure represents an accumulation factor in which the principal is accruing with no interest



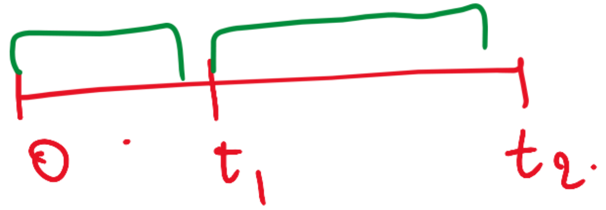
D. A step increasing accumulation factor. This is an accumulating factor in which interest is not accruing continuously but is accruing in discrete segments with no interest accruing between interest payment dates.

3 Fund Value

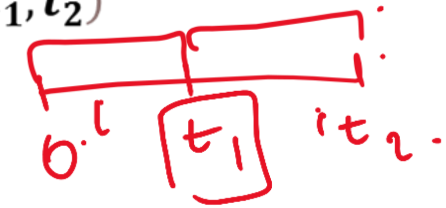
The **fund value is the total amount an investment currently holds**, including the capital invested and the interest it has earned to date (Accumulated value).

For instance, if 100 is invested at t = 0 in an investment giving 6% interest p.a., then the fund value at time t = 1 is 106 i.e. (100 + 6% of 100).

4 Principle of Consistency



- For $t_1 \le t_2$, we define $A(t_1, t_2)$ to be the accumulation at time t_2 of a unit investment made at time t_1 for a term of (t_2 - t_1)
- The quantity $A(t_1, t_2)$ is called an accumulation factor for a term of (t_2 - t_1). For an investment of sum C at time t_1, the accumulation at time t_2 is = C. $A(t_1, t_2)$
- We define A(t, t) = 1 for all t, reflecting that the accumulation factor must be unity over zero time.
- The proceeds at time t_2 will be $A(t_0, t_2)$ if one invests at time t_0 for term (t_2 - t_0), or $A(t_0, t_1)$ x $A(t_1, t_2)$ if one invests at time t_0 for term (t_1 - t_0) and then, at time t_1, reinvests the proceeds for term (t_2 - t_1) . In a consistent market, these proceeds should not depend on the course of action taken by the investor.
- Accordingly, we say that under the principle of consistency: $\boldsymbol{A(t_0, t_2) = A(t_0, t_1) \times A(t_1, t_2)}$



Question

An investment of Rs. 10,000 is made into a fund at time t = 0. The fund develops the following balances over the next 4 years:

t	A(t)
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

If Rs. 5000 is invested at time t = 2, under the same interest environment, find the accumulated value of the Rs. 5000 at time t = 4.

Question

Arjun has saved Rs. 10,000 in the past 4 months. He plans to invest this in an investment fund. The investment provides returns at a rate of 8% per annum. The investment grows over time t, according to the following investment factor:

$A(t_1, t_2) = (1 + \mathrm{i})^{(t_2 - t_1)}$ for $t_1 < t_2$.

i) Calculate the Accumulated Value of Arjun's investment:

a) After 10 years

b) After 15 years

ii) Calculate the Accumulating factor between time 10 and 15.

5 Effective Rate of Interest

- **General notation –** ***i***

- **Definition:** The effective rate of interest i is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.

- **Alternative definition:** The effective rate of interest i is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

- Interest is paid *once per measurement period*. This will be later contrasted with "nominal" rates of interest, in which interest is paid more frequently than once per measurement period.

- Effective rate of interest i_n, in terms of accumulation:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \quad for\; n = 1,2,3 \ldots$$

$\boxed{100}$ $\boxed{104}$

0 4

$$\frac{104-100}{100} = 4\%$$

Question

An investment of Rs. 10,000 is made into a fund at time t = 0. The fund develops the following balances over the next 4 years:

t	A(t)
0	10000.00
1	10600.00
2	11130.00
3	11575.20
4	12153.96

Find the effective rate of interest for each of the 4 years.

6 Simple Interest

P×R×T.

Simple interest does not compound, meaning that an investor will only gain the principal and the interest on the principal, and not interest on interest

Interest calculated on the principal portion of a loan, investment

Under simple interest, the **interest earned every year remains the same.** Int.

AF
Av.
P, I, ...

Amount under the fund grows linearly.

9%

100 × 9%

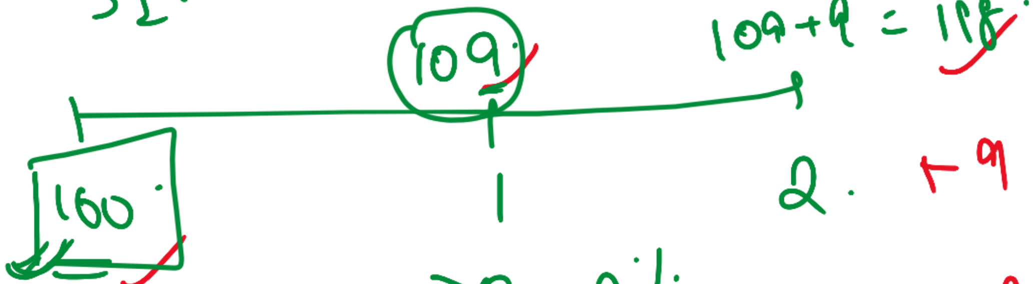
9% × 100

9% pa

9% × 100

9

SI.



109 × 9%

2. +9

+9

6.1 Example

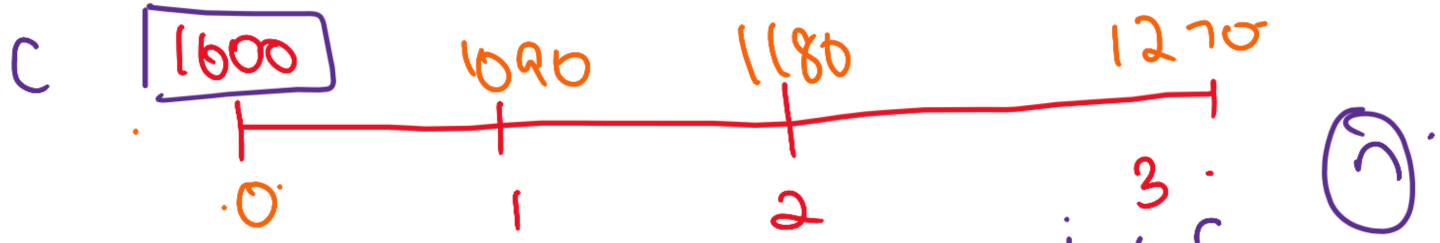
The current rate of interest quoted by a bank on an investment is **9% simple interest per annum**. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.
Therefore, balance at the end of first year = 1000(1 + 0.09) = 1090.

Now under the simple interest system the interest amount of 90 will not be reinvested.
Hence the principal amount remains 1000 as it is.
Balance at the end of second year = 1000(1 + 2x0.09) = 1180.

Similarly, balance at the end of third year = 1000 (1 + 3x0.09) = 1270.
Thus, Accumulated value at the end of 3 years is 1270.



9% SI p.a

Int amt for the 1st year = $\underset{i}{9\%} \times \underset{c}{1000} = 90$

AV at time 1 = $1000 + 90 = 1090$ $\qquad c + ci = c(1+i)$

Int amt for the 2nd year = $\underset{i}{9\%} \times \underset{c}{1000} = 90$

AV at time 2 = $1000 + 90 + 90 = 1180$

$$C + Ci + Ci = c(1+2i)$$

$$\boxed{AV = c(1+ni)}$$

Int 3rd year = $9\% \times 1000 = 90$

AV at 3 = $\underset{c}{1000} + \underset{ci}{90} + \underset{ci}{90} + \underset{ci}{90} = 1270 = c(1+3i)$

$(1+ni)$ Accumulation factor.

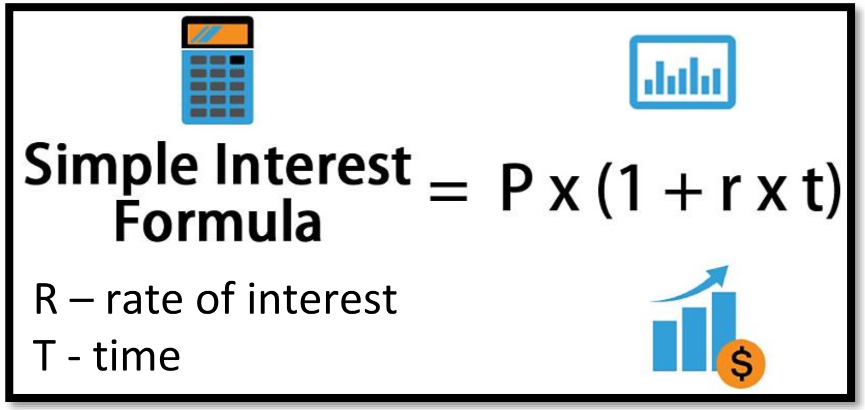
$$\frac{(1+9\%)}{8]} = (1+i)^3$$

$$(1+9\%)^{\frac{1}{3}} - 1 = i$$

6.2 Simple Interest - Generalisation

Consider:
Amount Invested – C
Simple interest rate – I

- Accumulated value at the end of the 1st period = C(1+i)
- Accumulated value at the end of the 2nd period = C(1+2i)
- And so on, the accumulated value at the end of the nth period = C(1+ni)



The **Accumulation factor** under simple interest system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$A(t_1, t_2) = (1 + (t_2 - t_1) * i)$ or
$A(n) = (1 + ni)$

$A(t_1, t_2) = (1 + (t_2 - t_1) i)$
$A(n) = (1 + ni)$

Question

1)
Amount Deposited = 10,000
Simple Interest = 7% pa

Accumulated amount after 3 years? $\boxed{12100}$

$C\times(1+ni)$

10,000

S.I.

$\boxed{7\%\ pa}$

$\boxed{10210}$

$\boxed{10,000}(1+3\times0.07)$

$10,000(1.21)$

$\boxed{12,100}$

$10,000\times7\%\times3$

Int.

2)
Amount Deposited = 15,000 Simple Interest = 5% pa

Compare how much the investor would have after 8 years if the money was:

a. Invested for 8 years $\boxed{21000} \rightarrow 15000\times(1+8\times5\%)$
b. Invested for 4 years, then immediately reinvested for further 4 years.

21,600

At $t=0$, Int = 15000

$C = 15000$

AV at time 4 $= C \times (1+4i)$

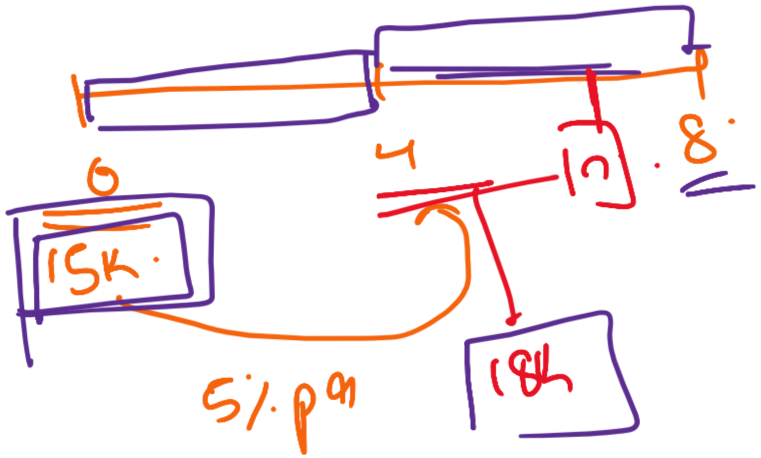
$= 15000(1+4\times 5\%)$

$= 18000$

AV at time 8 $= C \times (1+4i)$

$= 18000\,(1+4\times 5\%)$

$= \boxed{21600}$



a) $15000 = C$

b)

$\underline{C\times(1+7i)} = A$, $A\cdot(1+i)$

7 Compound Interest

Int =

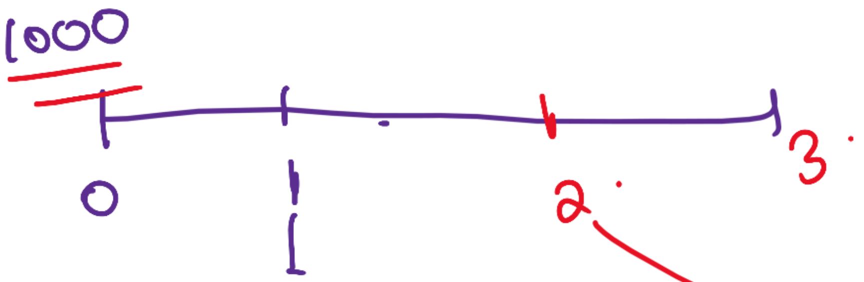
1000

0 1 2 3 4

Clearly, it is more advantageous for the investor to invest the interest earned so far, in order to earn more ahead.

The word **"compound" refers to the process of interest being reinvested to earn additional interest**. With compound interest, the total investment of principal and interest earned to date is kept invested at all times.

9% CI pa.



$9\% \times 1000$
$= 90$
$AV = 1090$

$9\% \times 1090$
$= 98.1$
$AV = 1090 + 98.1$
$= 1188.1$

$9\% \times 1188.1$
$= 106.929$
$AV = 1295.029$

7.1 Example

CI

The current rate of interest quoted by a bank on an investment is **9% compound interest per annum**. Suraj makes an investment of Rs. 1000. Assuming that there are no other transactions, determine the accumulated value just after interest is credited at the end of 3 years.

1000

0 1 2 3

$1000 \times 9\% = 90$

$1000 + 90$

$= 1090$

Suraj invests 1000 at the start.

During the first year his investment will grow at the rate of 9%.
Therefore, balance at the end of first year = $\underbrace{1000}_{C} + \underbrace{1000 \times 0.09}_{C \times i} = \underbrace{1000(1.09)}_{C(1+i)} = 1090$

Under compound interest this balance is reinvested and earns interest in the second year, producing a balance of = 1090 + 1090 x 0.09 = 1090(1.09) = 1000(1.09)^2 = 1188.10 at the end of the 2nd year.

$C(1+i) + C(1+i) \times i \quad C(1+i)(1+i) = C(1+i)^2$

The balance at the end of the third year will be
= 1188.10 +1188.10 x 0.09 = (1188.10)(1.09) = 1000(1.09)^3 = 1295.03

$C(1+i)^2 + C(1+i)^2 \times i \quad C(1+i)^2(1+i) = C(1+i)^3$

$\boxed{AV = C(1+i)^n}$

$AF = (1+i)^n$

$C(1+i)^1$

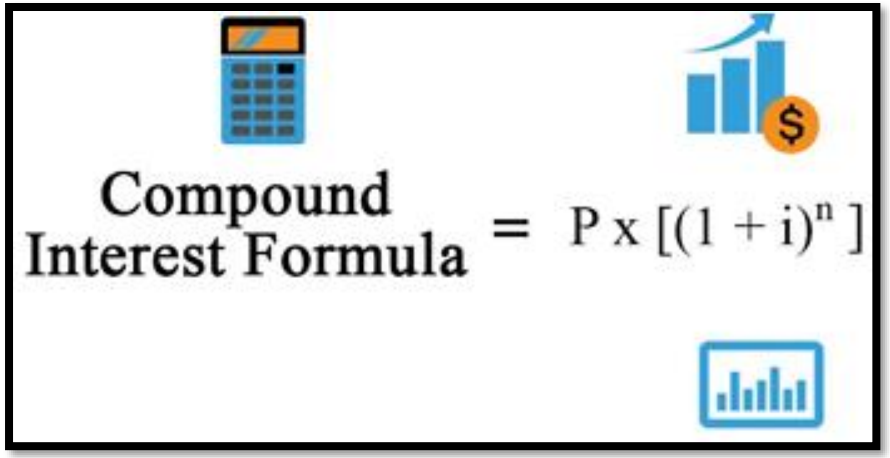
$C(1+i)^2$

$C(1+i)^3$

7.2 Compound Interest - Generalisation

Consider:
Amount Invested – C
Compound interest rate – i

- Accumulated value at the end of the 1st period = C + Ci = C(1 + i)
- Accumulated value at the end of the 2nd period = C(1 + i) + C(1 + i) x i = $C(1+i)^2$
- And so on the account will continue to grow by a factor of (1 + i) per year, resulting in a balance of $\boxed{C(1+i)^{n}}$ at the end of n years



C

The **Accumulation factor** under compound interest system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$$A(t_1, t_2) = (1+i)^{(t_2 - t_1)} \text{ or}$$
$$A(n) = (1+i)^{n}$$

$A(n) = (1+ni)$

$A(t_1, t_2) = (1+i)^{(t_2-t_1)}.$

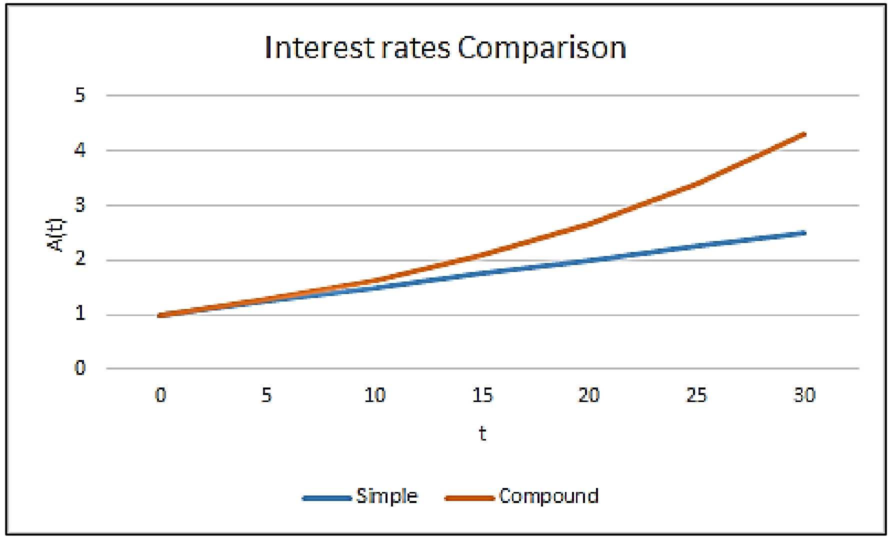
$A(n) = (1+i)^{n}.$

7.3 Simple Interest vs Compound Interest

SIP.

1

- It is clear that simple and compound interest produces the same result over one measurement period.
- As seen above, the accumulated value at the end of first year is 1090 under both simple interest and compound interest.
- Over a longer period, compound interest produces a larger accumulated value than simple interest while the opposite is true over a shorter period.



SI
AV
AF

CI.
AV
AF

Watch – Difference between SI & CI
https://www.youtube.com/watch?v=5wpsLW5JEms

SI:

9% pa

3 years

0		1		2		3
	90		90		90	
1000		$1000(1+9\%) = \boxed{1090}$		$1000(1+2\times 9\%) = 1180$		$1000(1+3\times 9\%) = 1270$

int amt.

$\boxed{\text{absolute amt}}$:

1090 − 1000

1180 − 1090

1270 − 1180

contd.

C = 1000

n = 0.25

i = 9% pa SI / CI.

SI :

$$1000(1+0.25\times 9\%) = \boxed{1022.5}$$

$$CI = 1000\times(1+9\%)^{0.25} = \boxed{1021.778}$$

CI. 9% pa.
3 years.

0	1	2	3
1000	$1000(1+9\%)$ = 1090	$1000(1+9\%)^2$ = 1188.1	$1000(1+9\%)^3$ = 1295.03

Interest per year: 90 (0 to 1), 98.1 (1 to 2), 106.92 (2 to 3).

CI : absolute amt does not stay constant.
ratio : stays constant

$t=1$: $\frac{1090}{1000} = 1.09$ | $t=2 = \frac{1188.1}{1090} = 1.09$ | $t=3$.

$IA_n > IA_{n-1} > IA_{n-2}$ ...

$C = 1000$ $n = 0.25$ $i = 9\%$ $i = 9\%$

SI =

CI =

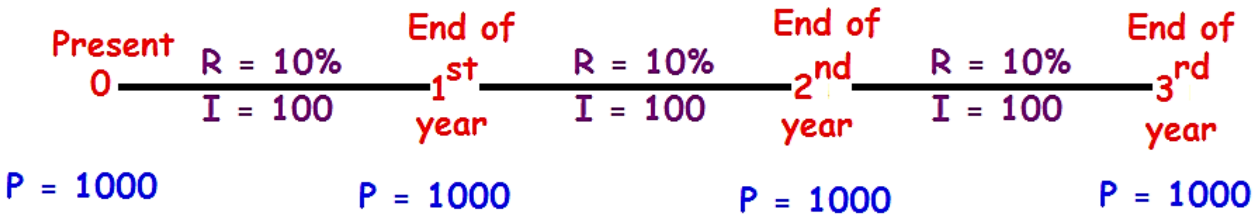
7.3 Simple Interest vs Compound Interest

2

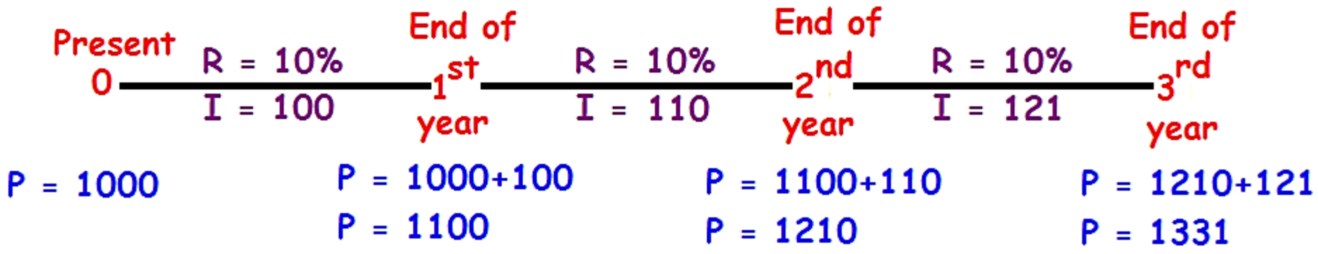
- Under simple interest, it is the **absolute amount of growth** that is **constant** over equal periods of time

- Under compound interest, it is the **relative rate of growth** that is **constant**

SIMPLE INTEREST :



COMPOUND INTEREST :



7.3 Simple Interest vs Compound Interest

3

- A constant rate of simple interest implies a decreasing effective rate of interest.

Let i be the constant rate of simple interest and let i_n be the effective rate of interest for the nth period. Then we have

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{(1+n.i)-(1+(n-1)i)}{(1+(n-1)i)} = \frac{i}{(1+(n-1)i)}$$ for n = 1,2,3,..., which is a decreasing function of n.

- In the case of compound interest, it can be shown that a constant rate of compound interest implies a constant effective rate of interest and, moreover, that the two are equal.

Let i be the constant rate of compound interest and let i_n be the effective rate of interest for the nth period. Then we have

$i_n = i$

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}} = (1+i)^{n-(n-1)} - 1 = (1+i) - 1 = i$$

which is independent of n. Thus, although defined differently, a rate of compound interest and the corresponding effective rate of interest are same.

1000 — 1090 — 1180 — 1270

0 — 1 — 2 — 3

$\boxed{SI = 9\%}$

3 years

$SI = 9\%$ p.a.

$i_{SI} = 9\%$ p.a.

Eff. rate of int

$$i_3 = \frac{1270 - 1180}{1180} = 7.62712\%$$

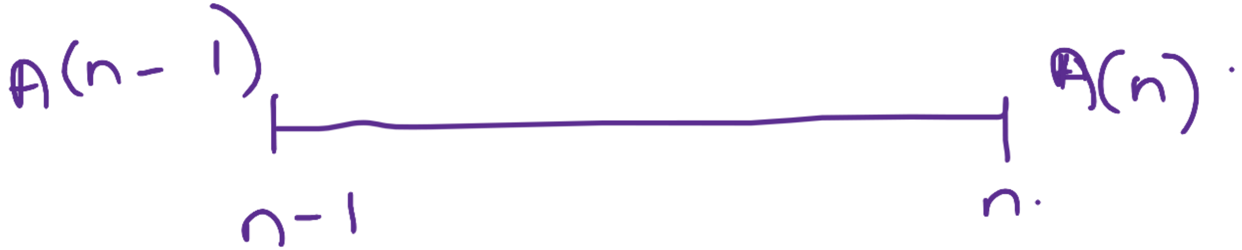
$$i_2 = 8.2569\% = \frac{1180 - 1090}{1090}$$

$$i_1 = 9\% = \frac{1090 - 1000}{1000}$$

Eff RI = $\dfrac{\text{Int amt}}{\text{Amt @ } n-1}$

1 → 2 → 3

9%   8.2569%   7.6271



$$A(n) = (1+i)^n$$

SI

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \leftarrow \text{ERI}$$

CI

$i_n =$ ______

$$i_n = \frac{(1+ni) - (1+(n-1)i)}{(1+(n-1)i)}$$

$$i_n = \frac{\cancel{1} + \cancel{ni} - \cancel{1} - \cancel{ni} + i}{(1+(n-1)i)}$$

$n \uparrow$

$i_n \downarrow$

simple int

$$i_n = \frac{i}{(1+(n-1)i)}$$

ERI

CI.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

$$i_n = \frac{A(n)}{A(n-1)} - \frac{A(n-1)}{A(n-1)}$$

$$i_n = \frac{(1+i)^n}{(1+i)^{n-1}} - 1$$

$$i_n = (1+i)^1 - 1 \qquad \boxed{i_n = i}$$

$A(n-1)$ $A(n)$

9% CI. $i_n = i$

1000	1090	1188.1	1295.03
0	1	2	3

$$i_3 = \frac{1295.03 - 1188.1}{1188.1}$$

$$i_3 = 9\%$$

7.3 Simple Interest vs Compound Interest

4

- Compound interest is used almost exclusively for financial transactions covering a period of one year or more and is often used for shorter term transactions as well.
- Simple interest is occasionally used for short-term transactions and as an approximation for compound interest over fractional periods.

AV , AF , ERI , SI

CI .

ERI .

SI CI .

Question

An investment of Rs. 10,000 in an account accumulates to Rs. 30,000 after 5 years.

1. State the accumulating factor A(0,5) or A(5).

2. ~~1.~~ Find the simple annual interest rate which would give the accumulation factor in part 1.

3. ~~1.~~ Find the annual compound interest rate which would give the accumulation factor in part 1.

1

$A(5) = 3 = 300\%$

2

$A(n) = (1+ni)$

$3 = (1+5i)$

$i = 40\%$

3

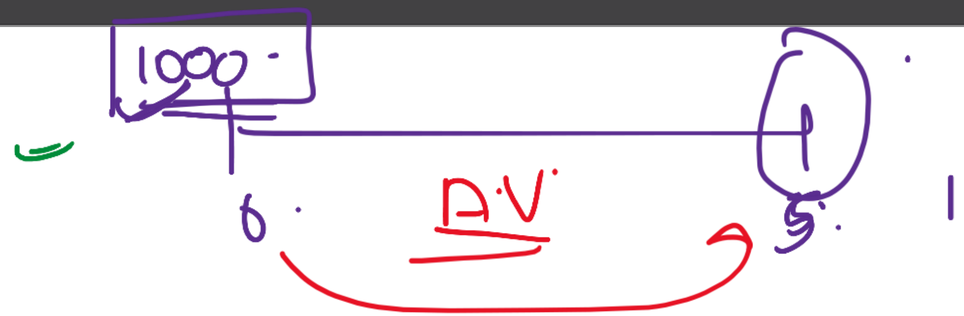
$A(5) = (1+i)^5$

$3 = (1+i)^5$

$3^{\frac{1}{5}} = (1+i)$

$i = 24.5731\%$

8 Present Value

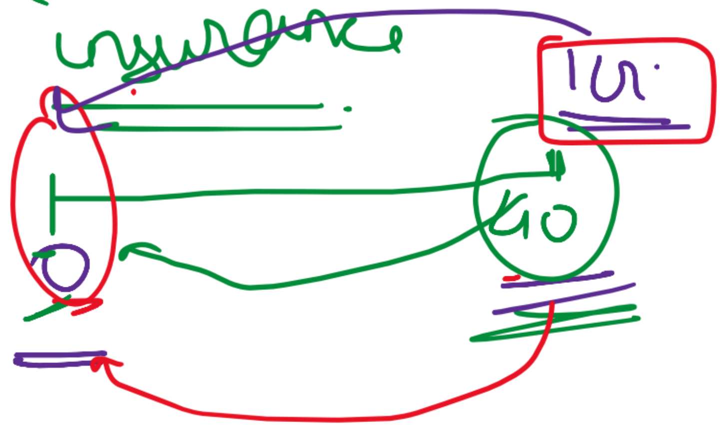


- Accumulation factors were introduced to quantify the growth of an initial investment as time moves forward. However, one can consider the situation in the opposite direction.
- As an actuary we are more interested in the 2nd situation.
- If one has a future liability of known amount at a known future time, how much should one invest now (at known interest rate) to cover this liability when it falls due?

i.e. what is the current value of the future liability taking time value of money (interest) into account?

AV > Principal

PV



8.1 Discounting Factor

$30{,}000 = 10{,}000 \times A(5)$

$30{,}000 \times \boxed{\frac{1}{A(5)}} = 10{,}000$

$V(t_1, t_2)$ - Present Value factor or Discounting factor; Gives the PV or discounted value for time t_1 to t_2 of an amount of 1 due at time t_2.

$30{,}000 \times V(5) = 10{,}000$

$A(t_1, t_2)$ - Accumulating factor; Gives the accumulation for time t_1 to t_2 of an investment of 1 at time t_1.

Discounting factor is the reciprocal of the accumulating factor

$$\boxed{V(t_1, t_2) = \frac{1}{A(t_1, t_2)}}$$

1

t_1 t_2

$A(t_1, t_2)$

$$\boxed{V(n) = \frac{1}{A(n)}}$$

1

t_1 t_2

$A(n) = \frac{1}{V(n)}$

$V(t_1, t_2) \Rightarrow$ PV factor / Discounting factor

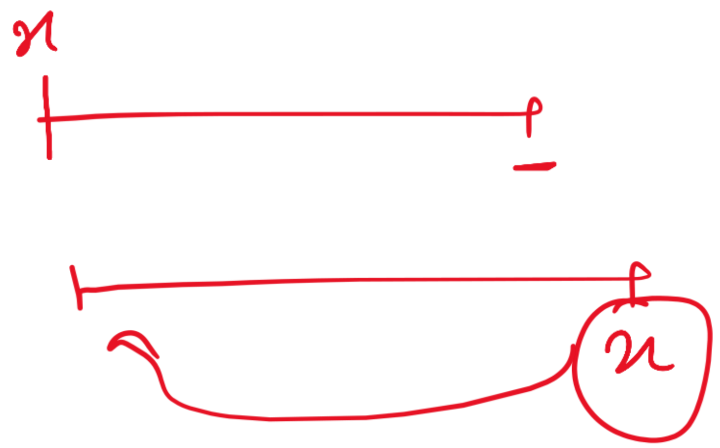
unknown

$100\, A(2) = 120$

$100 = 120 \times \frac{1}{A(2)}$ $\frac{1}{AV}$

SI: $A(n) = 1 + ni$.

$$V(n) = \left(\frac{1}{1+ni}\right)$$



CI: $A(n) = (1+i)^n$

$$\boxed{V(n) = \frac{1}{(1+i)^n}}$$

$$(1+i)^{-n}.$$

$$\boxed{V(n) = v^n.}$$

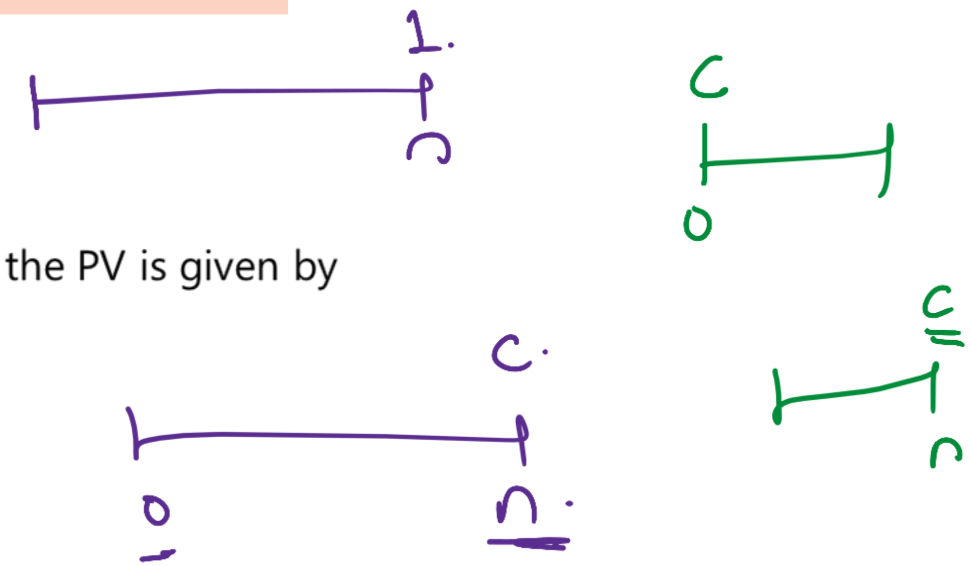
$$\boxed{v = (1+i)^{-1}}$$

where $v = \dfrac{1}{(1+i)}$.

8.2 Present Value – Simple Interest

- Discounting Factor: $V(t) = \frac{1}{1+it}$
- For a balance of amount C at the end of t time periods, the PV is given by

PV = $C.V(t) = C\left(\frac{1}{1+it}\right)$ for t>0



$C \times \frac{1}{(1+ni)}$

$C \times A(n)$ $\quad$ $C \times V(n)$

Question

1. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 5000 at the end of 5 years.

Assume Simple interest. 9%

5000

0 5

SI = 9% pa.

3448.27586.

Shift mode.

Fix 9.

9.

2. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 1270 at the end of 3 years.

Assume Simple interest.

$$PV = C \times \frac{1}{(1+ni)} = 5000 \times \frac{1}{(1+5\times 9\%)}.$$

$$= \boxed{3448.27586.}$$

999.99 ≈ 1000.

≈ 1000.

8.3 Present Value – Compound Interest

- We need to determine how much a person must invest initially so that the balance will be 1 at the end of one period. The answer is $(1 + \mathrm{i})^{-1}$, since this amount will accumulate to 1 at the end of one period.
- Discounting Factor: $V(t) = \frac{1}{(1+\mathrm{i})^t} = (1 + \mathrm{i})^{-t}$
- Let $v = (1 + \mathrm{i})^{-1}$
- For a balance of amount C at the end of t time periods, the PV is given by

  $$PV = C.V(t) = C.\frac{1}{(1+\mathrm{i})^t} = Cv^t \text{ for } t>0$$

- Thus, Cv^t is the present value at time 0 of an amount C due at time t when an investment grows according to compound interest. This means Cv^t is the amount that should be invested at 0 to grow to C at time t, and the present value factor v acts as 'compound present value' factor in determining present value.

$C_0 (1+i)^{-n}$.

Cv^n.

Question

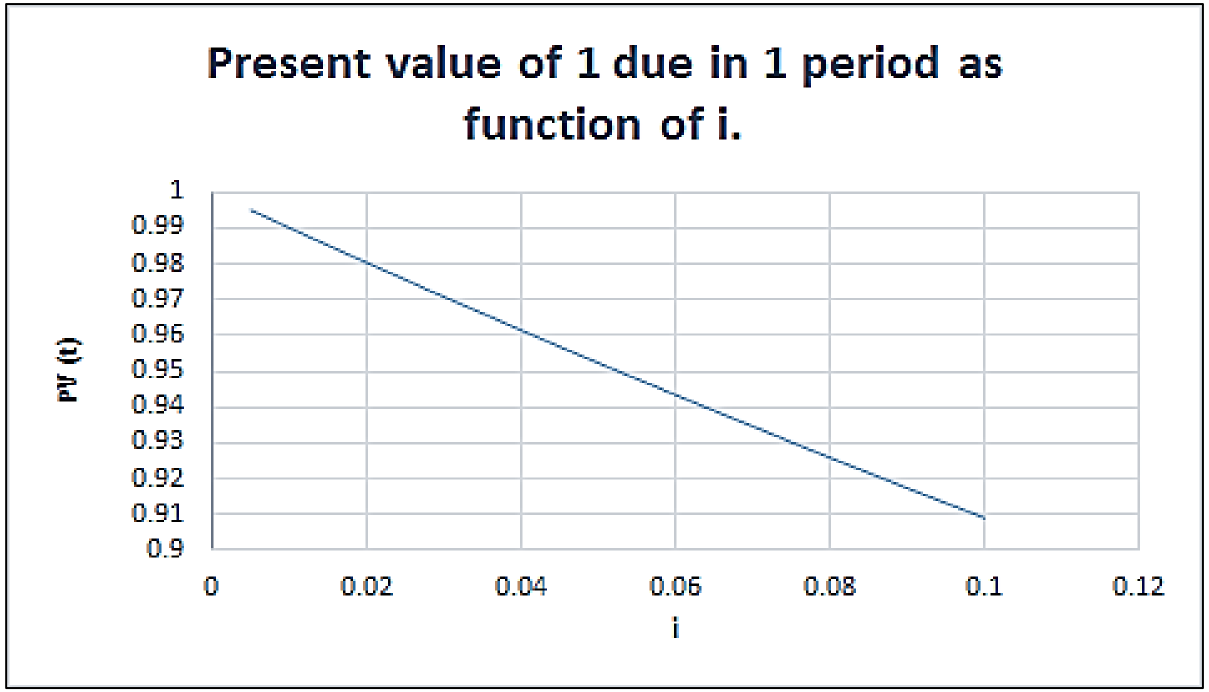
1. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 5000 at the end of 5 years.

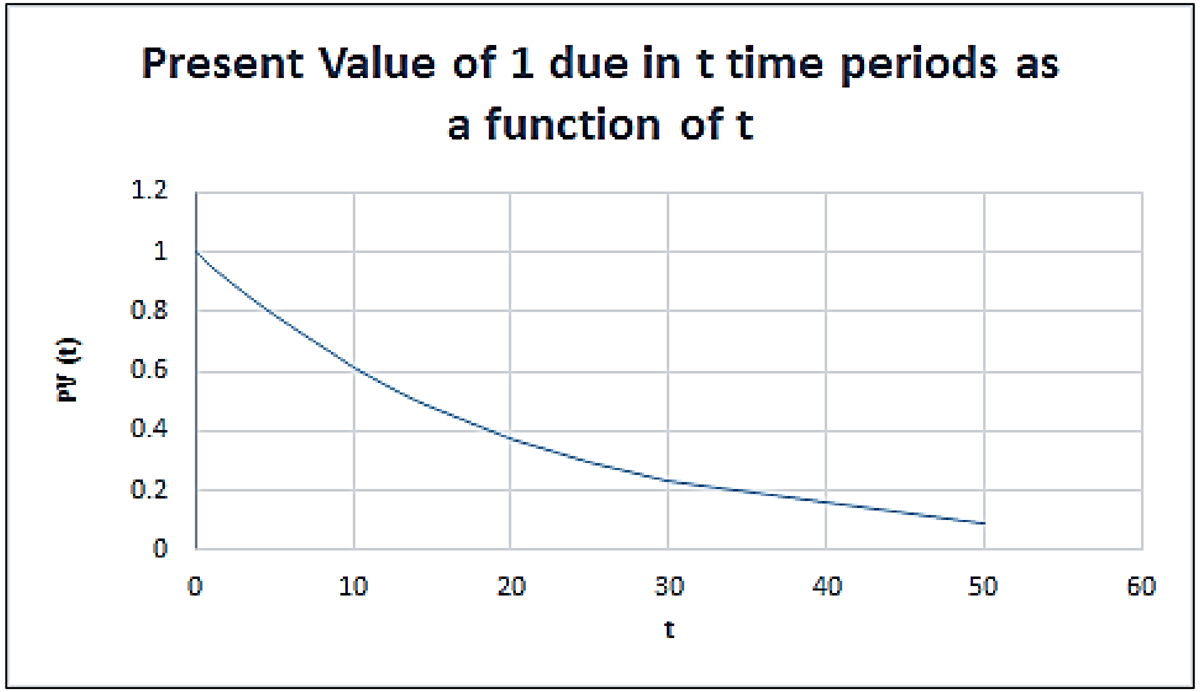
Assume Compound interest. - $\boxed{3249.65\cdot69.}$ $PV = 5000 \cdot \frac{1}{(1+9\%)^5} =$

2. Find the amount which must be invested at 9% p.a. in order to accumulate Rs. 1270 at the end of 3 years.

Assume Compound interest. $\boxed{980.6730}$. $1270 \cdot \frac{1}{(1+9\%)^3} = .$

8.4 Simple Interest vs Compound Interest





JAQS

Question

Mr. Dev wants to invest a sufficient amount in a fund in order that the accumulated value will be 1 Crore on his retirement date in 25 years.

1yr.

25.

Dev considers two options. He can invest in Equity Mutual Fund (E.M. Fund), which invests in the stock market. E.M. Fund has averaged an annual compound rate of return of 19.5% since its inception 30 years ago, although its annual growth has been as low as 2% and as high as 38%.The E.M. Fund provides no guarantees as to its future performance.

EMF = 19.5% pa.

Dev's other option is to invest in a zero-coupon bond or stripped bond, with a guaranteed effective annual rate of interest of $\boxed{11.5\%}$ until its maturity date in 25 years.

ERI = 11.5% pa.

~~1.~~ What amount must Mr. Dev invest if he chooses E.M. Fund and assumes that the average annual growth rate will continue for another 25 years? 116359.6397 — bond.

$1\times 10^7 \cdot \left(\frac{1}{1+19.5\%}\right)^{25}$ =

2 1. What amount must he invest if he opts for the stripped bond investment?

= 657852.1758 =

3. ~~1.~~ What minimum effective annual rate is needed over the 25 years in order for an investment of Rs. 2,50,000 to accumulate to Dev's target of 1 Crore? 15.89972% / 27.08

4. ~~1.~~ How many years are needed for Dev to reach $\boxed{\text{1 Crore}}$ if he invests the amount found in part (a) in the stripped bond?

40.91386 ≈ 41 years

1cr.

n

250,000 $\qquad\qquad\qquad\qquad$ 1×10^7.

0 $\qquad\qquad$ i $\qquad\qquad$ 25.

$$250{,}000\, A(25) = 1 \times 10^7.$$

$$A(25) = \frac{1 \times 10^7}{250{,}000}.$$

$$\frac{1}{(1+i)^{25}} = \frac{10000000}{250{,}000}.$$

$$i = \underline{15.8992\%}.$$

41 6359.6397. $\qquad\qquad$?n.

0 $\qquad\qquad\qquad$ 25.

bond = 11.5% pa

$$n = \underline{\qquad}.$$

$$116359.6397 = 10000000 \times \left(\frac{1}{1+11.5\%}\right)^n$$

$$\frac{116359.6397}{1 \times 10^7} = \left(\frac{1}{1+11.5\%}\right)^n$$

$$\boxed{85.94 = (1+11.5\%)^n}$$

$$\ln 85.94 = n \ln(1+11.5\%)$$

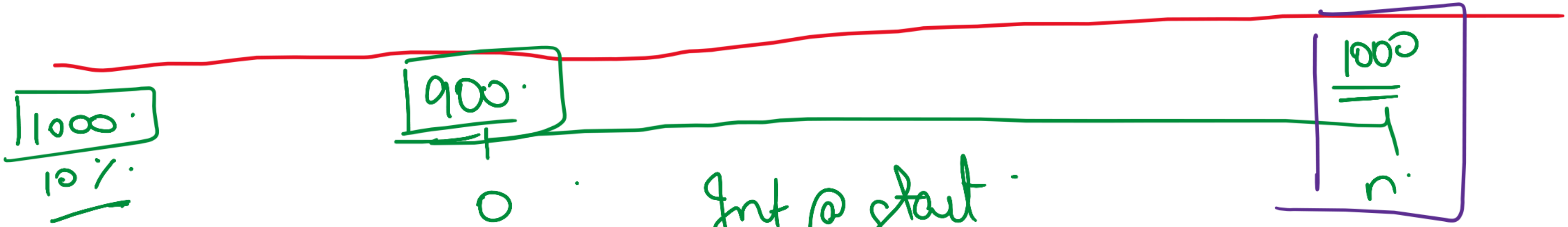
$$n = \frac{\ln 85.94}{\ln(1+11.5\%)} = \underline{40.9138}$$

1st way

$$i = \frac{\text{Amt of int @ end}}{\text{Amt @ } n-1}$$

$\boxed{\text{Int @ end}}$

$$\frac{100}{1000}$$



Int @ start

$$\boxed{\text{effective discount rate}} = \frac{\text{Amt of int @ start}}{\text{Amt @ end}} = \frac{100}{1000}$$

8.4

Interest Types – Arrears & Advance

Arrears = end

Advance = start.

Interest type can be divided on following basis:

a) For a time period, when is the interest charged? start/end

b) Interest amount is defined in what terms? = Amt @ n-1 / Amt @ n

100 110 0

- **Interest paid in arrears (Interest rate)** i
  - Interest amount paid/charged at the end of the period.
  - corresponding interest rate is the ratio of the amount of interest paid for the period to the amount of principal at the start of the period
- **Interest paid in advance (discount rate)** d
  - Interest amount paid/charged at the start of the period.
  - corresponding interest rate is the ratio of the amount of interest paid for the period to the amount due at the end of the period

6% x100

94 0

Int @ start

9 Effective Rate of Discount

- General notation – *d*
- **Definition:** The effective rate of discount d is the ratio of the amount of interest (sometimes called the "amount of discount" or just "discount") earned during the period to the amount due at the end of the period.

$\frac{\text{Amt of int}}{\text{Amt @ n.}}$

- **Alternative definition:** The effective rate of discount d is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the start of the period.
- Interest is paid *once per measurement period.*

1000.
800.
20%

- Effective rate of interest, d_n in terms of accumulation:

$$d_n = \frac{\boxed{A(n) - A(n-1)}}{A(n)} \quad for\ n = 1,2,3 \ldots$$

- Effective annual interest measures growth on the basis of the initially invested amount, whereas effective annual discount measures growth on the basis of the year-end accumulated amount. Either measure can be used in the analysis of a financial transaction

ERI ERD
SI CI SD CD

ERI

$\boxed{i = 6\%}$

100 → 106

0 → 1

Int @ end.

$$d = \frac{106 - 100}{106} \qquad \frac{\text{Amt of int}}{\text{Amt @ } n}$$

$$d = 5.6604\%$$

$$\boxed{i = 6\%} \longleftrightarrow \boxed{d = 5.6604\%}$$

ERD.

$\boxed{d = 6\%}$

94 → 100

0 → 1

Int @ start.

$$i = \frac{100 - 94}{94} = \frac{\text{Amt of int}}{\text{Amt @ } n-1}$$

$$i = 6.3829\%$$

$$i = \frac{\text{Amt of int}}{\text{Amt @ start}} \qquad d = \frac{\text{Amt of int}}{\text{Amt @ end}}$$

Illustration

100 1 year $i = 6\%$ 100 100 + 6 0 1 = 106 A

If A goes to a bank and borrows Rs.100 for one year at an effective rate of interest of 6%, then the bank will give A Rs.100.
At the end of the year, A will repay the bank the original loan of Rs.100, plus interest of Rs.6, or a total of Rs.106.

6
$6\% \times 100$

However, if A borrows Rs.100 for one year at an effective rate of discount of 6%, then the bank will collect its interest of 6% in advance and will give A only Rs. 94. At the end of the year, A will repay Rs.100.

94 100 ~~100~~ 0 1

d
i

Thus, it is clear that an effective rate of interest of 6% is not the same as an effective rate of discount of 6%. In the above example, A paid Rs.6 interest in both cases. However, in the case of interest paid at the end of the year, A had the use of Rs.100 for the year, while in the case of interest paid at the beginning of the year, A had the use of only Rs.94 for the year.

A
6

Note: Some readers may find the use of the word “paid” in connection with rates of discount somewhat confusing, since the borrower does not directly “pay” the interest as with rates of interest. However, the net result of deducting the interest in advance is no different than if the full amount is borrowed and then the borrower immediately pays the interest.

9.1 Example

1000 → 1 year. → 10%
100

Suraj borrows 1000 for one year at a quoted rate of 10% with interest payable in advance, the 10% is applied to the loan amount of 1000, resulting in an amount of interest of 100 for the year. The interest is paid at the time the loan is made.

Suraj receives the loan amount of 1000 and must immediately pay the lender 100, the amount of interest on the loan. One year later he must repay the loan amount of 1000. The net effect is that Suraj receives 900 and repays 1000 one year later.

The effective annual rate of interest on this transaction is $\frac{100}{900}$ = 0. 1111, or 11.11%.

0   1

900   1000

This 10% payable in advance is called the rate of discount for the transaction.

100
900

$\boxed{i = 11.11\%}$

The rate of discount is the rate used to calculate the amount by which the year end value is reduced to determine the present value.

$\boxed{d = 10\%}$

The effective annual rate of discount is another way of describing investment growth in a financial transaction. In the example just considered we see that an effective annual interest rate of 11.11% is equivalent to an effective annual discount rate of 10%, since both describe the same transaction.

10 Equivalent Rates

Assume that a person borrows 1 at an effective rate of discount d.

Then, in effect, the original principal is 1 - d and the amount of interest (discount) is d. However, from the basic definition of i as the ratio of the amount of interest to the principal, we obtain:

$$i = \frac{d}{1-d}$$

$$d = \frac{i}{1+i}$$

$i = \frac{d}{1-d}$

$d = \frac{i}{1+i}$

$d = i \times \frac{1}{1+i}$

i and d

$d = \frac{6}{100} = 6\%$

$i = \frac{6}{94} = 6.3829\%$

Another important relationship between i, v and d is that; $\boxed{\mathbf{d = i.v}}$

This relationship has an interesting verbal interpretation. Interest earned on an investment of 1 paid at the beginning of the period is *d*. Interest earned on an investment of 1 paid at the end of the period is *i*. Therefore, if we discount *i* from the end of the period to the beginning of the period with the discount factor v, we obtain *d*.

$$v = \frac{1}{1+i}$$

$$v \quad \frac{1}{1+i}$$

$$PV \text{ of } C_1 = C \times \frac{1}{(1+i)^n}$$

$$= C \times v^n.$$

$$d = \frac{i}{1+i}.$$

$$d = i \times \frac{1}{(1+i)} = iv.$$

$i = \frac{\text{Int}}{\text{Amt @ st}}$ | $d = \frac{\text{Int}}{\text{Amt @ end}}$

$i \to d$

$1-d$ ——— 1

0 ——— 1

d

$$\boxed{i = \frac{1-(1-d)}{1-d} = \frac{d}{1-d}}$$

$d = 10\%$

$i = 11.11\%$

$d \to i$

1 ——— $\boxed{1+i}$

0 ——— 1

$$d = \frac{i}{1+i} \quad \frac{1}{10\%}$$

$\cancel{i}$ | 1 | 11.11% | $\frac{1}{1+i}$

$$d = \frac{1+i-1}{1+i}$$

$$\boxed{d = \frac{i}{1+i}}$$

(a) ERI = $\dfrac{\text{Int @ end}}{\text{Amt @ start}}$.

SI : $AV = C(1+ni)$

CI $AV = C \cdot (1+i)^n$.

(b) ERD = $\dfrac{\text{Int @ start}}{\text{Amt @ end}}$.

SD :

CD :

$$i = \frac{d}{1-d}$$

$$d = \frac{i}{1+i}.$$

AF | N f
AV | PV
i | d
SI CI | SD CD

11 Simple Discount

Simple discount does not compound.

Under simple discount, the interest earned every year remains the same.

Amount under the fund grows linearly.

Simple discount
Compound discount
nominal rates of int

11.1 Example

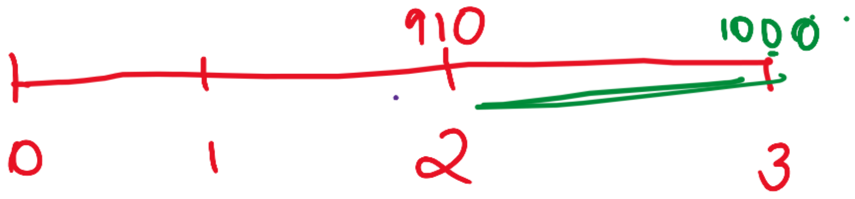
Table

i $\quad a_{\overline{n|}}$ $\quad$ IFoA

Kunal borrows 1000 for three years at a quoted rate of 9% with interest payable in advance, the 9% is applied to the loan amount of 1000. (simple discount)

Find the present value of this financial transaction at time 0, 1 and 2.

$d = 9\%$ simple rate of dis



$d = 9\%$ pa

$c - d \cdot c = c(1-d)$

$1000 \to c$

$9\% \to d$

94 $\quad$ 6% $\quad$ 100

$100 - 6\% \times 100$

$PV_2 = 1000 - 9\% \times 1000$

$= 1000 - 90$

$PV_2 = 910$

0 1 2 3

730 820 910 1000

$c(1-2d) \leftarrow c(1-d) - d \times c$

$PV_1 = 910 - 9\% \times \boxed{1000}$

$= 910 - 90$

$PV_1 = 820$

$c(1-2d) - d \times c = c(1-3d)$

$PV_1 = 820 - 9\% \times 1000$

$= 820 - 90$

$= 730$

c

0 n

0 1 2 3

$PV_0 = c \times (1 - nd)$. PVC / Disc F

$d \to -ve$

$ \to +ve$

11.2 Simple Discount - Generalisation

d AV

Consider:
Amount due – C for n years
Simple discount rate – d

$PV = C \times (1 - nd)$.

$AV = \frac{1}{PV} = C \times \frac{1}{(1-nd)}$

- Present value 1 year before i.e. time n-1 = C(1-d)
- Present value 2 years before i.e. time n-2 = C(1-2d)
- And so on, the present value at time 0 = C(1-nd)

The **Present Value/ Discounting factor** under simple discount system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$1 - nd$

$1 - (t_2 - t_1) \cdot d$.

$$V(t_1, t_2) = (1 - (t_2 - t_1) * d)$$

Thus:

$$A(t_1, t_2) = \frac{1}{V(t_1, t_2)} = \frac{1}{1 - (t_2 - t_1)d}$$

Question

Amount Deposited = 10,000
Simple discount = 7% pa

Accumulated amount after $\boxed{3 \text{ years?}}$

$\boxed{i} \rightarrow AV$ $\quad d \rightarrow AV \times, PV$ ✓

10,000

0 3

(SI) $d = 7\%$ pa

$AV_3 = 10000 \times (1 - 0.07 \times 3)^{-1}$

$AV_3 = 12658.2785$

12 Compound Discount

Calculations under compound discount are analogous to those under compound interest system. We understand the process through an example.

Kunal borrows 1000 for three years at a quoted rate of 9% with interest payable in advance under compound discounting.

Find the present value of this financial transaction at time 0, 1 and 2.

$d = 9\%$ pa

CD

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 753.571 & 828.1 & 910 & \boxed{1000} \end{array}$$

$c - d \times c = c(1-d)$

$PV_2 = 1000 - 9\% \times 1000$

$= 1000 - 90$

$PV_2 = 910$

$c(1-d) - d \times c(1-d)$

$PV_1 = 910 - 9\% \times 910$

$= 910 - 81.9$

$PV_1 = 828.1$

$c(1-d)(1-d)$

$c(1-d)^2$

$PV_0 = 828.1 - 9\% \times 828.1$

$= 828.1 - 74.529$

$PV_0 = 753.571$

$c(1-d)^2 - d \times c(1-d)^2$

$c(1-d)^2(1-d)$

$c(1-d)^3$

$PV_0 = c(1-d)^n$

$\hookrightarrow$ Disc factor

$PV_0 = AV(1-d)^n$

95.

100.

TB → 91 days.

12.1 Compound Discount - Generalisation

Consider:
Amount due – C after n years
Compound discount rate – d

- Present value 1 year before i.e. time n-1 = C - Cd = C(1 - d)
- Present value 2 years before i.e. time n-2 = C(1 - d) + C(1 - d) x d = $C(1-d)^2$
- And so the present value at time 0 will be $C(1-d)^n$

The **Present Value/ Discounting factor** under compound discount system for a period from t_1 to t_2, where $t_1 < t_2$ is:

$$V(t_1, t_2) = (1-d)^{(t_2-t_1)}$$

$t_2 - t_1$.

Thus:

$$A(t_1, t_2) = \frac{1}{V(t_1, t_2)} = \frac{1}{(1-d)^{(t_2-t_1)}}$$

Question

Amount Deposited = 10,000
Compound discount = 7% pa

Accumulated amount after 3 years?

Int payable in advance

discount ⇒ d

PV · AF = AV

C: $\boxed{10,000}$ at 0, 3

$d = 7$ pa

d → i

$$AV_3 = 10,000 \cdot (1 - 0.07)^{-3}$$

$$= 12432.2906$$

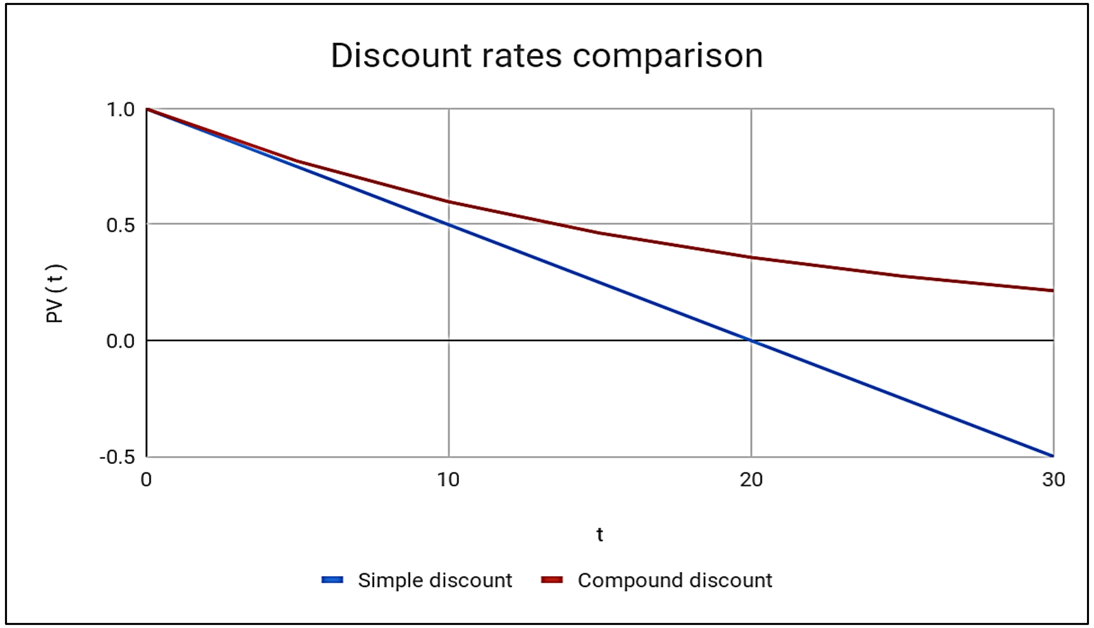
$$AV: \boxed{10,000 \cdot (1+i)^3 =}$$

13 Useful Insights

(0→n)

1. A constant rate of simple interest implies a decreasing effective rate of interest, as the period of investment increases, while a constant rate of simple discount implies an increasing effective rate of discount (and interest). (n→0)

2. Simple and compound discount produce the same result over one measurement period. Over a longer period, simple discount produces a **smaller** present value than compound discount, while the opposite is true over a shorter period.



comp dis

0	1	2	3
$\boxed{753.571}$	828.1	910	1000

$d = 9\%$ pa

simple dis

0	1	2	3
$\boxed{730}$	820	910	1000

Const SI → decreasing eff RI (0 → n)

Const SD → increasing ERD (n → 0)

Simple dis

0	1	2	3
730	820	910	1000

$d = 10.9756\%$ $\quad d = 9.8901\%$ $\quad d = 9\%$

$$= \frac{910 - 820}{910}$$

SD: $\boxed{d = 9\%\ pa}$

$$d = \frac{\text{Amt of int}}{\text{Amt @ end}}$$

$$i = \frac{\text{Amt of int}}{\text{Amt @ start}}$$

14 Nominal Rates of Interest

Interest rate sheet of a leading Indian bank showing varying interest rates:

100

$i^{(4)} = 4\%$

$\frac{i^{(4)}}{4} = 1\%$

INTEREST RATES ON DEPOSITS effective from 23rd September 2020.		
Maturity Periods	Regular	Senior Citizen
7 - 14 days	2.50%	3.00%
15 - 30 days	2.50%	3.00%
31 - 90 days	3.00%	3.50%
91 - 179 days	3.90%	4.40%
180 days	4.60%	5.10%
270 days	4.60%	5.10%
365 days	4.70%	5.20%
2 years	4.90%	5.40%
3 years	4.90%	5.40%
4 years	4.75%	5.25%
5 years	4.50%	5.00%

NRI.

15 days : $\boxed{2.5\%}$ per annum

1 year = $\boxed{4.70\%}$

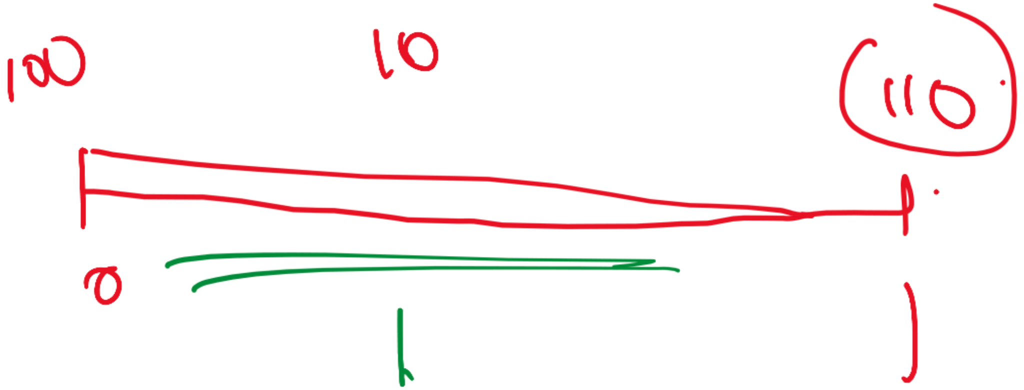
24

$\boxed{i^{(2)}}$ = 4.60% pa

$i^{(1)}$ = eff ROI

nominal int rate

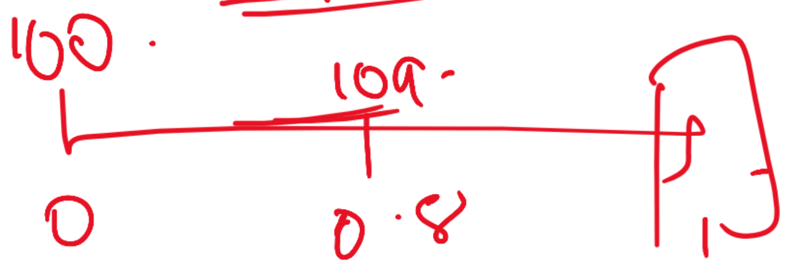
Source: https://www.kotak.com/en/rates/interest-rates.html

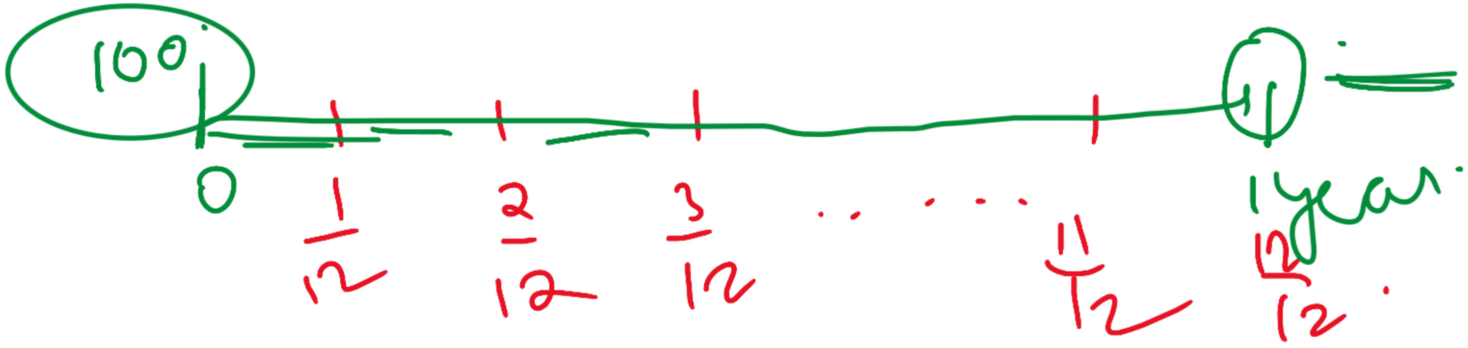


$\boxed{i = ERI}$

Int is paid only once per measurement period

9 % pa ERI

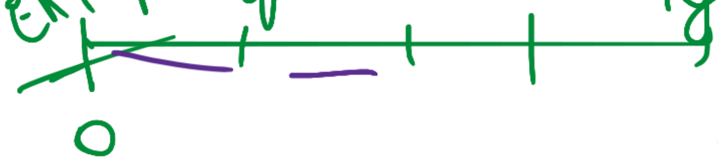




ER } → per month

14 Nominal Rates of Interest

3% ER per qtr



12% pa
Comp qua

- The term "effective" is used for rates of interest and discount in which interest is paid once per measurement period, either at the end of the period or at the beginning of the period, as the case may be.

int disc

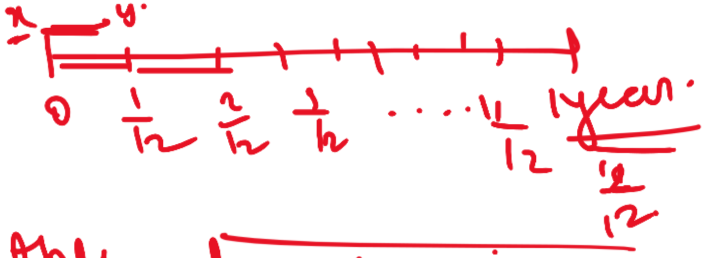
- In real world, we come across situations where interest is paid more frequently than once per measurement period. Rates of interest and discount in these cases are called "nominal."

- These interest rates are quoted for the entire period they are actually in practice credited more than once per measurement period (i.e nominally) to make an overall effective rate as mentioned for the entire period.

convertible
compounded

$\frac{12\%}{12} = 1\%$ 12% pa

12% pa comp monthly



Terms used in practice:

nominal int rate

1. Payable **e.g.** 12% p.a. payable monthly
2. Convertible **e.g.** 24% p.a. convertible quarterly
3. Compounded **e.g.** 6% p.a. compounded two-monthly

monthly rate of int = 1% eff rate of int per month.

Note: The three terms "payable," "compounded," and "convertible" are often used interchangeably.

14.1 Example

ER 1 pa

100 — 108 1 year

12% int rate

$\frac{12\%}{4}$

- Interest rate – 8% p.a. convertible quarterly (Nominal rate of interest)
- Quarterly interest rate – 2% (effective quarterly rate/ effective rate of interest per quarter)
- Effective annual interest rate – 8.2432% p.a. (Actual annual rate of interest)

12% pa rate comp monthly

1%

nominal rate of int / rate of int compounded pthly = $i^{(p)}$

ERI per pth period = $\frac{i^{(p)}}{p}$

$i^{(0.5)}$ / $i^{(12)}$ / $i^{(4)}$ / $i^{(2)}$

two-yearly monthly quarterly semi-annual

$(1+i)^n$ 12% p.

12% pa convertible mthly

$i^{(12)}$

$\frac{i^{(12)}}{12}$

8%

2% 2% 2% 2%

0 1q 2q 3q 1 year

100 102 104.04 106.12 4q.

108.24

$AV_{1q} = 100 \times (1+2\%)^1 = 102$

$AV_{2q} = 102 \cdot (1+2\%) = 104.04$

$AV_{3q} = 104.04\,(1+2\%) = 106.1208$

$AV_{4q} = 106.1208\,(1+2\%) = 108.2432$

$AV = (1+2\%)^4 \times 100$ — NRI

$= (1+8.2432\%)^1 \times 100$ — ERI

$\boxed{= 8.2432\%}$

$\boxed{8\%\ pa}$ convertible qtrly

$i^{(4)} = 8\%$ pa.

$\frac{i^{(p)}}{p}$

$\frac{i^{(4)}}{4} = 2\%$ eff rate int per qtr

$i = \frac{108.2432 - 100}{100}$

$i = 8.2432\%$

$$AF = \left(1 + \boxed{2\%}\right)^4 = (1+i)^1$$

$$= \left(1 + \frac{8\%}{4}\right)^4$$

ERI per month

$$\frac{\boxed{i^{(12)}}}{12}$$

$$AF = \left(1 + \boxed{\frac{i^{(p)}}{p}}\right)^{p \times n}$$

NRI comp pthly.

$$i^{(12)}$$

$$AF = (1+i)^n$$

$$\boxed{12}$$

$$\left(1 + \frac{i^{(p)}}{p}\right)^{n \times p} = (1+i)^n$$

$$\left(1 + \boxed{\frac{8\%}{4}}\right)^4 = (1 + 8.2432\%)^1$$

$i \rightarrow$ pa

14.1 Notation

$i^{(12)}$

$12\% = i^{(12)} \rightarrow$ nominal rate of int conv mthly

$1\% = \frac{i^{(12)}}{12} \rightarrow$ ER per month

Interest rate – 8% p.a. convertible quarterly (Nominal rate of interest) : $i^{(4)}$

Quarterly interest rate – 2% (effective quarterly rate/ effective rate of interest per quarter) : $\frac{i^{(4)}}{4}$

Effective annual interest rate – 8.2432% p.a. (Actual annual rate of interest): i

General:

- Nominal interest rate – convertible/ compounded/ payable pthly: $i^{(p)}$
- Effective pthly interest rate: $\frac{i^{(p)}}{p}$
- Effective annual interest rate: i

$\frac{i^{(p)}}{p}$ i

$\frac{i^{(1)}}{1} = i$

Note: The symbol for a nominal rate of interest payable p times per period is $i^{(p)}$ where p is a positive integer > 1. By a nominal rate of interest $i^{(p)}$ we mean a rate payable pthly, i.e. the rate of interest is $\frac{i^{(p)}}{p}$ for each pth of a period and not $i^{(p)}$.

8% 2%

Question

1)
Given: effective rate of interest:10% p.a.
Find: Nominal annual interest rate compounded semi-annually

$i = 10\%$ pa

100 — 1 year — 110

$p = 2$

$i^{(2)}$

$\dfrac{i^{(2)}}{2}$

ERI for each semi annual period

2)
Given: Nominal annual interest rate convertible four-monthly: 8%
a) Find: effective annual interest rate

$i^{(3)}$

ERI per four-month period

$p = 3$

$i^{(3)} = 8\%$ pa

$i = 8.21523\%$ pa

$\dfrac{i^{(3)}}{3}$

$\dfrac{8\%}{3}$

8% pa compounded qtly

$\dfrac{i^{(4)}}{4}$

8% pa

0 1 2 3

b) Find nominal annual int rate convertible monthly

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = (1+i)$$

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = (1+10\%)$$

$$\left(1+\frac{i^{(2)}}{2}\right) = (1+10\%)^{\frac{1}{2}}$$

$$i^{(2)} = 2\left[(1+10\%)^{\frac{1}{2}}-1\right] = 9.7618\%$$ pa compounded semi-annually.

14.2 Important Formulas

$$(1+\mathrm{i}) = \left(1+\frac{\mathrm{i}^{(p)}}{p}\right)^{p}$$

$$i = \left(1+\frac{\mathrm{i}^{(p)}}{p}\right)^{p} - 1$$

$$\mathrm{i}^{(P)} = p\left[(1+\mathrm{i})^{1/p} - 1\right]$$

$$\left(1+\frac{i^{(p)}}{p}\right)^{p} = (1+i)$$

$$p\left[(1+i)^{\frac{1}{p}} - 1\right]$$

$i^{(p)}$

p

$$\left(1+\text{Qtrly int rate}\right)^{4} = (1+i)$$

12% pa compounded $\boxed{\text{mthly}}$.

$i^{(12)}$

1% eff ROI. $\frac{i^{(12)}}{12}$ $\frac{i^{(4)}}{p}$ 12% $\frac{12\%}{12}$

Mathematical proof

$$(1 + \mathrm{i}) = \left(1 + \frac{\mathrm{i}^{(p)}}{p}\right)^{p}$$

Done in OneNote.

Question

1)
Investment: Rs. 5000 at time 0
Nominal interest rate payable monthly: 12% p.a.
Calculate fund value after 5 years

2)
Due: Rs. 10000 at time 5 years
Nominal interest rate convertible quarterly: 10% p.a.
Calculate the amount to be invested at time 0

5000 ? x

0 5

$i^{(12)} = 12\%$ pa

60

$5000 \times A(5) = x$

$5000 \times (1+i)^5 = x$

$\left(1+\frac{i^{(12)}}{12}\right)^{12} = (1+i)$

6102.709429

$5000 \times \left(1+\frac{i^{(12)}}{12}\right)^{60}$

$= 9083.4835$

$$PV_0 = 10{,}000 \cdot v(5)$$

$$= 10{,}000 \cdot \left(1 + \frac{10\%}{4}\right)^{-20}$$

$$= 6102.709$$

$i^{(4)} = 10\%$ pa

$$AF = \left(1 + \frac{i^{(4)}}{4}\right)^{20}$$

$$DF = \frac{1}{AF} = \left(1 + \frac{i^{(4)}}{4}\right)^{-20}$$

Question

Sneha inherits Rs. 100,000. She deposits it in a 5 year certificate of deposit paying 6% nominal interest compounded monthly and the interest remains on deposit. At the end of 5 years, Sneha decides to renew her CD for another 5 years at the then current nominal interest rate of 7.5% compounded quarterly. At the time her 2nd CD matures, what is her investment worth?

Done in OneNote.

14.3 Notation

Discount rate – 6% p.a. convertible quarterly (Nominal rate of discount) : $d^{(4)}$
Quarterly discount rate – 1.5% (effective quarterly discount rate/ effective rate of discount per quarter) : $\frac{d^{(4)}}{4}$
Effective annual discount rate – 5.8663% p.a. (Actual annual rate of discount): d

General:

- Nominal discount rate – convertible/ compounded/ payable pthly: $d^{(p)}$
- Effective pthly discount rate: $\frac{d^{(p)}}{p}$
- Effective annual discount rate: d

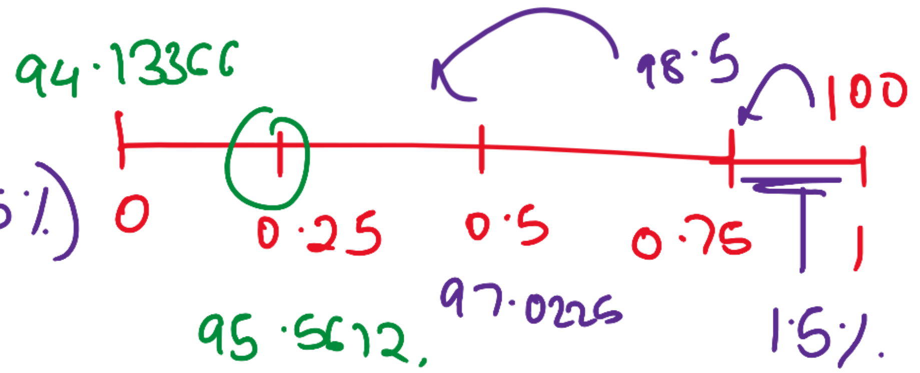
Note: The symbol for a nominal rate of discount payable p times per period is $d^{(p)}$ where p is a positive integer > 1. By a nominal rate of discount $d^{(p)}$ we mean a rate payable pthly, i.e. the rate of discount is $\frac{d^{(p)}}{p}$ for each pth of a period and not $d^{(p)}$.

$\boxed{d^{(4)} = 6\%}$ pa

$$\left(1-d\right) = \left(1-\frac{d^{(p)}}{p}\right)^{p}$$

ERD pa

$$\frac{d^{(4)}}{4} = \frac{6\%}{4} = 1.5\% \text{ eff per qtr.}$$



$$PV_{0.75} = 100 \cdot \left(1 - \frac{d^{(4)}}{4}\right) = 100 \cdot (1 - 1.5\%)$$
$$= 98.5$$

$$PV_{0.5} = 98.5 \cdot (1 - 1.5\%) = 97.0225$$

$$97.0225 \cdot (1 - 1.5\%) = 95.5672 \quad \Big| \quad 95.5672 \cdot (1 - 1.5\%) = 94.1336$$

Question

1)
Given: effective rate of discount: 5% p.a.
Find: Nominal annual discount rate compounded semi-annually

2)
Given: Nominal annual discount rate convertible four-monthly: 10%
Find: effective annual discount rate

14.3 Important Formulas

$$(1 - d) = \left(1 - \frac{d^{(p)}}{p}\right)^{p}$$

PV PV factors.

$$d = 1 - \left(1 - \frac{d^{(p)}}{p}\right)^{p}$$

$$d^{(P)} = p\left[1 - (1 - d)^{1/p}\right]$$

100 5 .

Question

1)
Investment: Rs. 5000 at time 0
Nominal discount rate payable monthly: 12% p.a.
Calculate fund value after 5 years

2)
Due: Rs. 10000 at time 5 years
Nominal discount rate convertible quarterly: 10% p.a.
Calculate the amount to be invested at time 0

$10,000 \times v(20)$

$10,000 \times \left(1 - \frac{d^{(4)}}{4}\right)^{20}$

5000 ?

0 5

$d^{(12)} = 12\%$ pa

two $\boxed{10,000}$

0 5

9138.15096

10% p.a

$d^{(4)}$

$5000 \times A(60) = 5000 \times \frac{1}{v(60)} = 5000 \times \frac{1}{\left(1 - \frac{d^{(12)}}{12}\right)^{60}}$

$= 9138.15096$

6026.8768

Interest

AV. Principal Interest.

AF.

0 t

0 t

PV

x

Discounting

PV factor.

Eff RoI.

$$i = \frac{\text{Amt of int}}{\text{Amt @ start}}$$

Eff RoD.

$$d = \frac{\text{Amt of int}}{\text{Amt @ end}}$$

$i = 6\%$

100 106

$d = 6\%$

100

94 100

ST

I

$AV = C \times (1 + ni)$

CI. $\longrightarrow$ $AV = C \times (1 + i)^n$.

$AV = \frac{1}{PV}$.

SD

CD $\longrightarrow$ $PV = C \times (1 - d)^n$.

$PV = C \times (1 - nd)$

$Af = C \times \frac{1}{(1 - nd)}$.

d

i

AV

PV

14.4 Useful Insight

pT.

It should be clear from general reasoning that with a given nominal annual rate of interest, the more often compounding takes place during the year, the larger the year-end accumulated value will be, so the larger the equivalent effective annual rate will be as well.

$i =$

$i^{(2)} =$

$i^{(4)} =$

$i^{(6)} =$

$i^{(12)} =$

i

i

i

i

$\boxed{i}$

$i = 12\%$ pa

p	$i^{(p)}$
1	12%
2	11.66%
4	11.4994%
8	11.4135%
12	11.38655%
52	11.3452%
365	11.3346%

$$(1+i) = \left(1 + \frac{i^{(p)}}{p}\right)^{p}$$

$$\left[(1+i)^{\frac{1}{p}} - 1\right] p = i^{(p)}$$

$$i^{(\infty)}$$

14.5 Example

Effective rate of interest = 12% p.a.
Find equivalent nominal annual interest rates for p = 1,2,3,4,6,8,12,52,365, ∞

p	
1	0.12
2	0.1166
3	0.1155
4	0.1149
6	0.1144
8	0.1141
12	0.1139
52	0.1135
365	0.113346
∞	0.113329

Highlights:

- More frequently the compounding takes place (i.e., as p increases), the smaller is the equivalent nominal annual rate
- The change is less significant, in going from monthly to weekly or even daily compounding, so there is a limit to the benefit of compounding.
- The limiting case $(p \to \infty)$ above is called continuous compounding and is related to the notions of force of interest and instantaneous growth rate of an investment

$$\boxed{i = 12\%\ pa}$$

$d^{(1)}$

$d^{(p)}$

p
1
2
4
8
12
52
365

$$d = \frac{i}{1+i}$$

$$(1-d) = \left(1 - \frac{d^{(p)}}{p}\right)^p$$

$$\left(\frac{1}{1+i}\right) = \left(1 - \frac{d^{(p)}}{p}\right)^p$$

$$\left(\frac{1}{1+i}\right)^{\frac{1}{p}} = 1 - \frac{d^{(p)}}{p}$$

$$\therefore \left[1 - \left(\frac{1}{1+i}\right)^{\frac{1}{p}}\right] p = d^{(p)}$$

14.5 Example

Effective rate of interest = 12% p.a.
Find equivalent nominal annual discount rates for p = 1,2,3,4,6,8,12,52,365, ∞

p	
1	0.107143
2	0.1102
3	0.1112
4	0.1117
6	0.1123
8	0.1125
12	0.1128
52	0.1132
365	0.11331
∞	0.113329

14.6 Useful Insight

For i > 0 and p > 1:

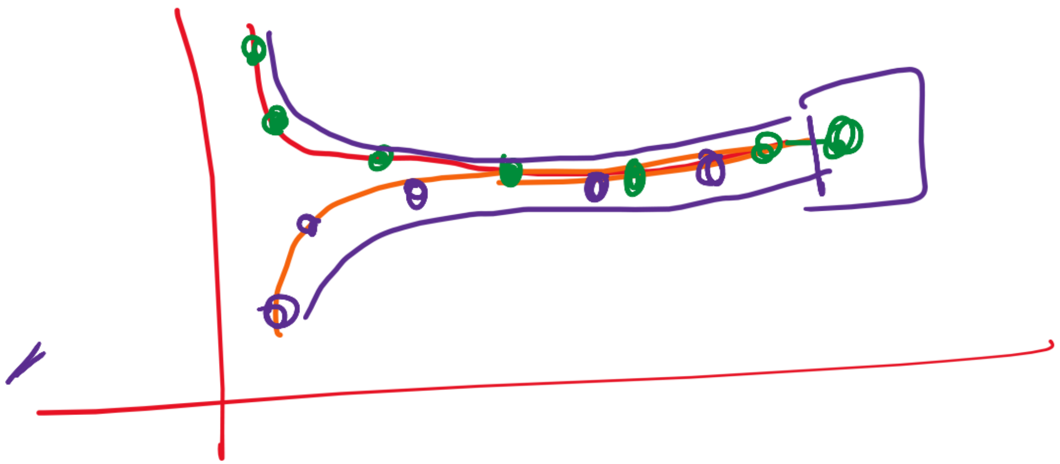
$$\mathbf{i} > \mathbf{i}^{(2)} > \mathbf{i}^{(4)} > \mathbf{i}^{(12)} > \boxed{\cdots} > \mathbf{d}^{(12)} > \mathbf{d}^{(4)} > \mathbf{d}^{(2)} > \mathbf{d}$$

limiting value

i

p ↑

i



15 Force of Interest

Effective rates of interest and discount measure interest over one measurement period
Nominal rates of interest and discount measure interest over pths of a measurement period

$i^{(12)}$

It is also important to be able to measure the intensity with which interest is operating at each moment of time, i.e. over infinitesimally small intervals of time. **This measure of interest at individual moments of time** is called the **force of interest**.

Force of int

Instantaneous growth in intt.

rate of int

Question

Amount invested at time 0 = 1000
The amount accumulated to 1050 at time 2 months

Calculate:

1. Monthly effective interest rate
2. Nominal annual interest rate convertible two monthly

15 Force of Interest

- Consider an investment of 1 unit at time 0.

- The interest rate earned by the investment for the $\frac{1}{p}$ year period from time t to $t + \frac{1}{p}$ time (i.e. pthly interest rate) is $\frac{A\left(t+\frac{1}{p}\right)-A(t)}{A(t)}$

- The nominal annual rate would be found by scaling up the pthly rate by a factor of p so that: $\mathrm{i}^{(p)} = p * \frac{A\left(t+\frac{1}{p}\right)-A(t)}{A(t)}$

- If p is increased, the time interval $[t, t + \frac{1}{p}]$ decreases, and we are focusing more and more closely on the investment performance during an interval of time immediately following time t. Taking the limit of $\mathrm{i}^{(p)}$ as $p \to \infty$, results in

$$\mathrm{i}^{(\infty)} = \lim_{p\to\infty} \mathrm{i}^{(p)} = \lim_{p\to\infty}\left[p \cdot \left(\frac{A\left(t+\frac{1}{p}\right)-A(t)}{A(t)}\right)\right]$$

15 Force of Interest

- Define h = $\frac{1}{p}$, so that as p → ∞, h → 0
- Solving it mathematically gives us: $i^{(\infty)} = \frac{A'(t)}{A(t)}$
- $i^{(\infty)}$ is a nominal annual interest rate compounded infinitely often or compounded continuously
- $i^{(\infty)}$ is also interpreted as the instantaneous rate of growth of the investment per unit invested at time point t and is called the force of interest at time t
- The notation for the force of interest at time is δ_t

$$\frac{A(t)}{A(0)} = e^{\int_0^t (\delta_S)\, dS}$$

$$AF = e^{\int_{t_1}^{t_2} \delta_s \, ds} \qquad DF = e^{-\int_{t_1}^{t_2} \delta_s \, ds}$$

$$A'(t) = A(t).\delta_t$$
$$A(t+h) - A(t) = A(t).\delta_t$$

- The differential expression A(t) . δ_t may be interpreted as the amount of interest earned on amount A(t) at exact time t due to the force of interest δ_t.

Question

1) Given $\boxed{\delta t = .08+.005t,}$ calculate the accumulated value $\boxed{\text{over five years}}$ of an investment of 1000 made at each of the following times:

(a) Time 0, and
(b) Time 2

$$\delta_t = 0.08 + 0.005t$$

2) Given δt = .05+.01t, calculate the present value at time 0 of two payments of 100 each to be paid at time t=2 and t=4.

A

$\delta_t = 0.08 + 0.005t$.

AV of 1000 at $t=5$

$= 1000 \times A(0,5)$

$= 1000 \times e^{\int_0^5 \delta_s \, ds}$

$= 1000 \, e^{\int_0^5 0.08 + 0.005t \, dt}$.

$= 1000 \, e^{\left[0.08t + \frac{0.005t^2}{2}\right]_0^5}$

$= 1588.0391$

$$e^{\left[0.08 \times 5 + \frac{0.005(5)^2}{2}\right] - \left[0.08 \times 0 + \frac{0.005(0)^2}{2}\right]}$$

$1000 \times A(2,7)$.

1169.459631

15.1 **Constant Force of Interest**

$\delta = 3\%$ $\delta = 8\%$

If the force of interest δ_n is constant with value δ from time 0 to time n then

$$A(n) = A(0) \cdot e^{\int_0^n \delta_t \, dt} = A(0) \cdot e^{\delta n} = A(0) \cdot \left(e^{\delta}\right)^n$$

identical to compound interest accumulation of the form A(n) = A(0).$(1+i)^n$

e^{δ}= 1+i ,or equivalently, δ = ln(1+i)

$$A(n) = e^{\int_0^n \delta_t dt}$$

Constant Force of Interest

- With compound interest and compound discount we have a constant force of interest $\delta_t = \delta$ for all $t \geq 0$

$$\delta_t = \frac{\partial}{\partial t} \ln\left[(1+i)^t\right] = \ln(1+i) = \delta$$

$$\delta_t = \frac{\partial}{\partial t} \ln\left[(1-d)^{-t}\right] = -\ln(1-d) = \delta$$

- The accumulation function becomes

$$a(n) = e^{\int_0^n \delta \, dt} = e^{\delta n} = (1+i)^n$$

$$A(n) = e^{\int_0^n \delta_t \, dt}$$

$$= e^{\delta \int_0^n 1 \, dt}$$

$$= e^{\delta [t]_0^n}$$

$$= e^{\delta [n - 0]}$$

$$A(n) = e^{\delta n}$$

$$A(t_1, t_2) = e^{\delta (t_2 - t_1)}$$

$$A(n) = e^{\delta n}$$

$$A(n) = (e^{\delta})^n$$

$$A(n) = (1+i)^n$$

$$1 + i = e^{\delta}$$

$$\boxed{i = e^{\delta} - 1}$$

$$\boxed{\ln(1+i) = \delta}$$

Question

Given i = 8%, calculate d, $d^{(12)}$, $i^{(4)}$ and δ

$i = 8\%$

$d =$

$d^{(12)} =$

$i^{(4)} =$

d = 7.4074%

d(12) = 7.671477%

i(4) = 7.7706%

δ = 7.6961%

$\delta =$

Summary

$$\ln(1+i) = \delta \qquad i = \frac{d}{1-d} \qquad d = \frac{i}{1+i}$$

$$A(0,n) = (1+ni) = (1+i)^n = \left(1+\frac{i^{(p)}}{p}\right)^{n\times p} = \frac{1}{(1-nd)} = \frac{1}{(1-d)^n}$$

$$e^{\delta}-1 = i$$

$$= \frac{1}{\left(1-\frac{d^{(p)}}{p}\right)^{n\times p}} = e^{\int_0^n \delta_t\,dt} = e^{\delta n}$$

$$V(0,n) = \frac{1}{(1+ni)} = \frac{1}{(1+i)^n} = \frac{1}{\left(1+\frac{i^{(p)}}{p}\right)^{n\times p}} = (1-nd) = (1-d)^n$$

$$= \left(1-\frac{d^{(p)}}{p}\right)^{n\times p} = e^{-\int_0^n \delta_t\,dt} = e^{-\delta n}$$

15.2 Formulas

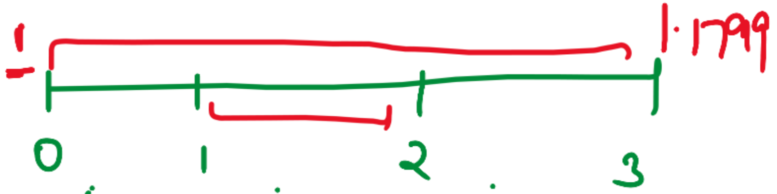
$$\mathbf{i} > \mathbf{i}^{(p)} > \boldsymbol{\delta} > \boldsymbol{d}^{(p)} > \boldsymbol{d}$$

$$i > i^{(p)} > \delta > d^{(p)} > d$$

Question

1. An investor in common stock measures investment returns annually using an effective rate of interest. The investor earns 15% during the first year, -5% during the second year, and 8% during the third year. Find the equivalent level effective rate of return over the three-year period

1. Rework the above example if the returns given are continuous measures, i.e. forces of interest, rather than effective rates.



AV of 1 at $t=3 = 1 \times (1+15\%)^1 \times (1-5\%)^1 \times (1+8\%)^1$

$= 1.1799$

$i_1 = 15\%$ $\quad i_2 = -5\%$ $\quad i_3 = 8\%$

$\delta_1 =$ $\quad \delta_2 =$ $\quad \delta_3 =$

$1.1799 = 1 \times (1+i)^3$

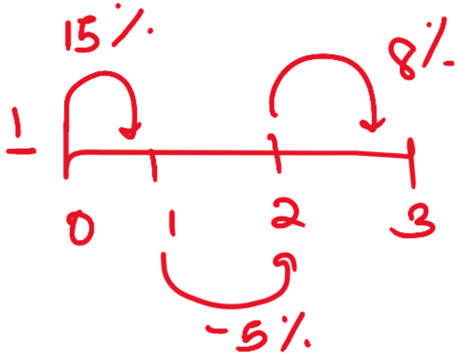
$i = 5.6691954\%$

$\boxed{6.18365\% \quad 6\%}$

$\delta_1 = 15\%$ $\qquad \delta_2 = -5\%$ $\qquad \delta_3 = 8\%$

$$AV \text{ of } 1 \text{ at } t = 3 = 1 \times e^{15\% \times 1} \times e^{-5\% \times 1} \times e^{8\% \times 1}$$

$$= \underline{\underline{1.197273}}$$



$$1.1972173 = 1 \times A(3)$$

$$1.1972173 = 1 \times e^{\delta \times 3}$$

$$\ln(1.1972173) = \delta \times 3$$

$$\delta = \frac{\ln(1.1972173)}{3}$$

$$\approx \boxed{\delta = 6\%}$$

Recap

- Interest – Compensation that a borrowers pays for use of capital.
- The fund value is the total amount an investment currently holds, including the capital invested and the interest it has earned to date (Accumulated value).
- Under the principle of consistency: $A(t_0, t_2) = A(t_0, t_1)\ x\ A(t_1, t_2)$
- The effective rate of interest i is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.
- Simple interest does not compound, meaning that an investor will only gain the principal and the interest on the principal, and not interest on interest.
- The word "compound" refers to the process of interest being reinvested to earn additional interest.

Recap

- V(t_1, t_2) - Present Value factor or Discounting factor; Gives the PV or discounted value for time t_1 to t_2 of an amount of 1 due at time t_2.

- The effective rate of discount d is the ratio of the amount of interest (sometimes called the "amount of discount" or just "discount") earned during the period to the amount due at the end of the period.

- Important relationship between i, v and d is that; d = i.v

- This measure of interest at individual moments of time is called the force of interest.

- If the force of interest δ_n is constant with value δ from time 0 to time n then
$A(n) = A(0) \cdot \mathrm{e}^{\int_0^n \delta_t \, \mathrm{d}t} = A(0) \cdot \mathrm{e}^{\delta_n} = A(0) \cdot \left(\mathrm{e}^{\delta}\right)^n$